UDC 531.36

ON STABILITY OF MOTION OF LINEAR SYSTEMS WITH RESPECT TO A PART OF VARIABLES

PMM Vol. 42, № 2, 1978, pp. 268-271

V.I. VOROTNIKOV and V.P. PROKOP'EV
(Sverdlovsk)

(Received July 4, 1977)

The question of reducing the problem of stability of motion with respect to a part of variables for linear systems with constant coefficients to a problem of stability of motion with respect to all variables for an auxiliary linear system which may be of dimension lower than that of the initial system, is considered.

1. Let us be given the following system of linear differential equations of perturbed motion with constant coefficients:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij}x_j \quad (i = 1, \ldots, n) \tag{1.1}$$

We shall consider the problem of stability of an unperturbed motion $x_i = 0$ $(i = 1, \ldots, n)$ relative to $x_1, \ldots, x_m$ $(m > 0,\ n = m + p,\ p > 0)$. We denote these variables by $y_i = x_i$ $(i = 1, \ldots, m)$, and the remaining variables by $z_j = x_{m+j}$ $(j = 1, \ldots, p)$[1, 2]. The equations of perturbed motion (1.1) now become

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}y_k + \sum_{l=1}^{p} b_{il}z_l \quad (i = 1, \ldots, m) \tag{1.2}$$

$$\frac{dz_j}{dt} = \sum_{k=1}^{m} c_{jk}y_k + \sum_{l=1}^{p} d_{jl}z_l \quad (j = 1, \ldots, p)$$

where a_{ik}, b_{il}, c_{jk} and d_{jl} are constants.

Let us transform the system (1.2) to a more suitable form by introducing new variables

$$\mu_i = \sum_{l=1}^{p} b_{il}z_l \quad (i = 1, \ldots, m) \tag{1.3}$$

and assuming that the first m_1 variables $(m_1 \leqslant m)$, $\mu_1, \ldots, \mu_{m_1}$ are linearly independent. Having introduced the new variables in this manner we find, that two cases are now possible:

In the first case the system (1.2) is reduced to the form

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}y_k + \sum_{l=1}^{m_1} \alpha_{il}\mu_l \quad (i = 1, \ldots, m) \tag{1.4}$$

$$\frac{d\mu_j}{dt} = \sum_{k=1}^{m} a_{jk}^*y_k + \sum_{l=1}^{m_1} \alpha_{jl}^*\mu_l \quad (j = 1, \ldots, m_1)$$

where only the linearly independent variables of (1.3) are chosen as μ_l and the behavior of the variables y_i with respect to which the stability of the unperturbed motion is being studied, is completely determined by the system (1.4). In what follows, we shall call such a system a μ-form system relative to the initial system (1.2).

In the second case the system (1.2) is not reduced to (1.4) and hence has the form

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}y_k + \sum_{l=1}^{m_1} \alpha_{il}\mu_l \quad (i = 1, \ldots, m) \tag{1.5}$$

$$\frac{d\mu_j}{dt} = \sum_{k=1}^{m} a_{jk}{}^*y_k + \sum_{l=1}^{m_1} \alpha_{jl}{}^*\mu_l + \mu_j^{(1)}, \quad \mu_j^{(1)} = \sum_{s=1}^{p} d_{js}{}^*z_s \ (j = 1, \ldots, m_1)$$

$$\frac{dz_r}{dt} = \sum_{k=1}^{m} c_{rk}y_k + \sum_{l=1}^{p} d_{rl}z_l \quad (r = 1, \ldots, p)$$

We introduce once again the new variables $\mu_j{}^{(1)}$ and assume that the first m_2 variables $(m_2 \leqslant m_1)$ $\mu_1{}^{(1)}, \ldots, \mu_{m_2}{}^{(1)}$ are linearly independent. Then the initial system (1.2) can be reduced to the form (1.4) or (1.5).

It can be shown that by repeating the above arguments we can always reduce system (1.2) to the μ-form. The order of the system obtained will not exceed the order of the initial system (1.2), and the eigenvalues of the μ-form system will belong to the set of eigenvalues of the system (1.2).

Indeed, the passage to the μ-form system is equivalent to replacing the variables $x = (y_1, \ldots, y_m, z_1, \ldots, z_p)$ where x is an n-dimensional vector, by the new variables $u = (y_1, \ldots, y_m, \mu_1, \ldots, \mu_h, \vartheta_1, \ldots, \vartheta_r)$ $(m + p = n = m + h + r)$, with the initial system (1.2) assuming such a form that the first $m + h$ equations do not contain ϑ_i $(i = 1, \ldots, r)$. In one particular case we can have $h = p$, i.e. the order of the μ-form system obtained can be equal to that of the initial system. Since we choose only the linearly independent variables from (1.3) and other similar expressions as μ_i, we can always perform such a passage to new variables. If we write the system (1.1) in the form $dx/dt = Px$ where P is a constant matrix, and consider the transformation $u = Lx$ to new variables (L is a constant nondegenerate matrix), the transformed system will have the form $du/dt = Qu$, $Q = LPL^{-1}$. The matrices P and Q are similar, therefore they have the same characteristic roots [3] i.e. the eigenvalues of the μ-form system belong to the set of the eigenvalues of the system (1.2).

Transformation of the initial system to the μ-form system enables us to consider, instead of the problem of stability of motion with respect to the variables y_i $(i = 1, \ldots, m)$ for (1.2), the problem of stability with respect to all the variables for the μ-form system. Obviously, the passage to the μ-form system is meaningful only when the μ-form system is of lower dimension than the initial system. Let us find the conditions under which the above statement is true. We write, for simplicity, the system (1.2) as

$$dy/dt = Ay + Bz, \quad dz/dt = Cy + Dz \tag{1.6}$$

where y and z are vectors of dimension m and p respectively, and A, B, C, D are constant matrices.

Assume that the initial system was reduced to the μ-form after introducing new variables β times. Then the new variables $\mu_i, \ldots, \mu_i^{(1)}, \ldots$ will represent the linearly independent columns of the matrix $K_\beta = (B', D'B', \ldots, D'^{\beta-1}B')$ where B' and D' are transposes of B and D.

Lemma 1. Let any column of the matrix D'^sB' be a linear combination of the columns of the matrix K_s. Then for any i $(i > s)$ the arbitrary column of the matrix D'^iB' is also a linear combination of the column of K_s.

Proof. Let $b_1, \ldots, b_m$ be the columns of the matrix B', i.e. $B' = (b_1, \ldots, b_m)$. Then $D'B' = (D'b_1, \ldots, D'b_m), \ldots, D'^jB' = (D'^jb_1, \ldots, D'^jb_m)$. Let

$$D'^sb_j = \sum_{k=1}^{m} \lambda_{jk}^{(0)} b_k + \sum_{k=1}^{m} \lambda_{jk}^{(1)} D'b_k + \ldots + \sum_{k=1}^{m} \lambda_{jk}^{(s-1)} D'^{s-1}b_k \quad (j = 1, \ldots, m) \qquad (1.7)$$

where $\lambda_{jk}^{(0)}, \lambda_{jk}^{(1)}, \ldots, \lambda_{jk}^{(s-1)}$ are constants. Then

$$D'^{s+1}b_j = D'(D'^sb_j) = \sum_{k=1}^{m} \lambda_{jk}^{(0)} D'b_k + \sum_{k=1}^{m} \lambda_{jk}^{(1)} D'^2b_k + \ldots + \sum_{k=1}^{m} \lambda_{jk}^{(s-1)} D'^sb_k \qquad (1.8)$$

Taking into account the fact that the equality (1.7) holds for the last term of (1.8), we find that any column of the matrix $D'^{s+1}B'$ is a linear combination of the columns of matrix K_s.

Lemma 2. The sufficient and necessary condition for the μ-form system for (1.2) to be of dimension N is, that the rank of the matrix K_p is $N - m$.

Proof. Necessity. Let the dimension of the μ-form system be equal to N. Using the method of reductio ad absurdum, we assume that the rank of $K_p = r \neq N - m$. Suppose that $r > N - m$. Then a number i $(i < p - 1)$ can be found such that the matrix K_{i+1} contains $N - m$ linearly independent columns and any column of the matrix $D'^{i+1}B'$ will, according to Lemma 1, be a linear combination of the columns of the matrix K_{i+1} since the dimension of the μ-form system will be equal to N. But this is impossible since in this case the matrix K_p will contain only $N - m$ linearly independent columns which contradicts the previous assumption. Therefore $r \leqslant N - m$. If $r < N - m$, then the dimension of the μ-form will not reach N, and this contradicts the condition of the lemma, therefore $r = N - m$.

Sufficiency. Let $\operatorname{rank} K_p = N - m$. According to [4] the columns of the matrix K_{N-m} contain $N - m$ linearly independent columns of the matrix K_p. Repeating the arguments expounded in the proof of necessity, we can now show that the dimension of the μ-form system is equal to N.

Corollary. The necessary and sufficient condition for the dimension of the μ-form system for (1.2) is, that $\operatorname{rank} K_p < p$.

2. Let us consider the stability of motion with respect to a part of the variables for the case of differential equations of perturbed motion with constant coefficients. Examples of the systems asymptotically stable with respect to a part of variables and unstable with respect to the other part of them were given in e.g. [5].

We shall base our criterion of asymptotic stability of the system (1.2) with respect to a part of the variables, on the reduction to the μ-form. The behavior of the variables $y_1, \ldots, y_m$ with respect to which the stability of the system is investigated are completely determined by the μ -form system for (1.2), hence the following theorem holds.

Theorem. The necessary and sufficient condition for the system (1.2) to be asymptotically stable with respect to the variables $y_1, \ldots, y_m$ is, that all eigenvalues of the μ -form system have negative real parts.

Corollary. Let the system (1.2) have m eigenvalues with negative real parts (the remaining eigenvalues with nonnegative real parts). The necessary and sufficient condition for the asymptotic stability of the system (1.2) with respect to variables $y_1, \ldots, y_m$ is, that the system has the form

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik} y_k, \quad i = 1, \ldots, m \tag{2.1}$$

$$\frac{dz_j}{dt} = \sum_{k=1}^{m} c_{jk} y_k + \sum_{l=1}^{p} d_{jl} z_l, \quad j = 1, \ldots, p, \quad p + m = n$$

and all roots of the equation $| a_{ik} - \delta_{ik}\lambda | = 0$ have negative real parts.

The sufficiency is obvious. To prove the necessity we assume the opposite, i.e. that the system (1.2) is not of the form (2.1). Then the dimension of the μ -form system for (1.2) is greater than m. The eigenvalues of the μ -form system belong to the set of the eigenvalues of the system (1.2), therefore by virtue of Theorem 1 we arrive at a contradiction with the asymptotic stability relative to $y_1, \ldots, y_m$.

The above corollary has been obtained earlier (*) by a different method.

Example. Let us consider the problem of asymptotic stability of the system

$$x_1^{\cdot} = -x_1 + x_2 - 2x_3, \quad x_2^{\cdot} = 4x_1 + x_2, \quad x_3^{\cdot} = 2x_1 + x_2 - x_3 \tag{2.2}$$

with respect to x_1. To do this we reduce the system (2.2) to the μ -form

$$x_1^{\cdot} = -x_1 + \mu, \quad \mu^{\cdot} = -\mu; \quad K_p = \begin{Vmatrix} 1 & -1 \\ -2 & 2 \end{Vmatrix} \tag{2.3}$$

The eigenvalues of the system (2.3) have negative real parts, consequently the unperturbed motion (2.2) is asymptotically stable with respect to x_1.

REFERENCES

1. Rumiantsev, V.V., On the stability of motion relative to a part of the variables. Vestn. MGU, Ser. matem., mekhan., astron., fiz., khim., No.4, 1957.
2. Oziraner, A.S. and Rumiantsev, V.V., The method of Liapunov functions in the stability problem for motion with respect to a part of the variables. PMM, Vol. 36, No. 2, 1972.
3. Kurosh, A.G., Lectures in General Algebra (English Translation) Pergamon Press, Book No. 10352, 1965.
4. Krasovskii, N.N., Theory of Control of Motion. Moscow, "Nauka", 1968.
5. Zubov, V.I., Mathematical Methods of Investigating Automatic Control Systems. Leningrad, Sudpromgiz, 1959.

Translated by L.K.

*) Peiffer K. La méthode direct de Liapounoff appliquée à l'étude de la stabilité partielle (Dissertation). Université Catholique de Louvain, Faculté des Sciences, 1968.